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Consensus of positive real systems cascaded with a single integrator

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SUMMARY

We study output consensus in a network of interconnected non-identical positive real systems cascaded with a single integrator. Assuming undirected, diffusive interconnections, sufficient conditions are provided to ensure output consensus for two cases: (1) the individual systems are weakly strictly positive real systems cascaded with a single integrator and (2) the individual systems are positive real systems cascaded with a single integrator. We illustrate our results with the example of a load frequency control network of synchronous generators.

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KEY WORDS: Consensus; output consensus; positive real systems; weakly strictly positive real systems; load frequency control.

1. INTRODUCTION

We study output consensus in a network of linear systems whose models are represented by the multiplication of positive real (PR - to be defined below) systems and a single integrator in the s-domain. To motivate our work, consider that such a network can be seen as a generalization of single-integrator consensus networks, which have been well-studied in the literature (see [1–4] and the references therein). Suppose we have N single-integrators $\dot{y}_i = u_i$, where $i = 1, \dots, N$. Further suppose these integrators are interconnected via their control inputs u_i according to the following diffusive output interconnection, $u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(y_j - y_i)$, where $a_{ij} > 0$ are interconnection coefficients. In this expression we denote the set of integrators that are connected to the i^{th} integrator as its "neighborhood" $\mathcal{N}_i$. Then we take the Laplace transform to obtain

$$Y_i(s) = \frac{1}{s} \sum_{j \in \mathcal{N}_i} a_{ij}(Y_j(s) - Y_i(s)),\ i = 1, \dots, N. \tag{1}$$

This is the typical description of a single-integrator consensus network, where the concern is that all output differences $y_i(t) - y_j(t)$ approach zero as t goes to infinity.

As a generalization of (1), we could consider the following network:

$$Y_i(s) = \frac{1}{s} \sum_{j \in \mathcal{N}_i} G_i(s) a_{ij}(Y_j(s) - Y_i(s)),\ i = 1, \dots, N, \tag{2}$$

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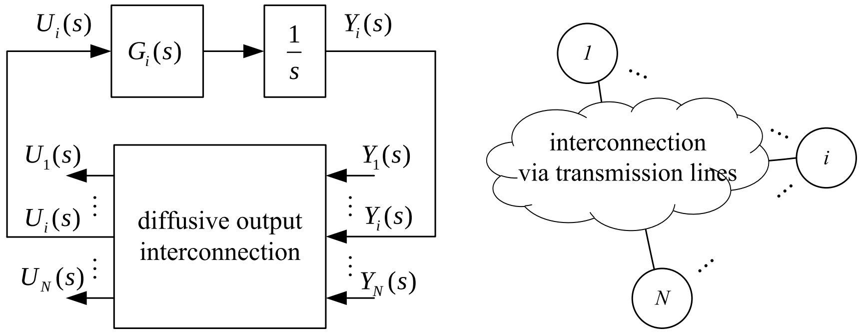

(a) Generalized single-integrator network.

(b) Diagram of a power grid.

Figure 1. Output diffusively interconnected consensus network.

which can be viewed as a single-integrator consensus network with dynamic interconnection coefficients $G_i(s)a_{ij}$. To reflect positivity of coefficients a_{ij} in (1), we assume that $G_i(s)$ are positive real. Figure 1a shows the block diagram of network (2). An alternate interpretation is that (2) can arise when a local controller is added to the i^{th} system in the standard consensus network (1).

A third interpretation of (2) is that this type of system arises as a natural model of a physical system. One such example is the load frequency control (LFC) network of an electrical power grid. Figure 1b shows a diagram of an electrical power grid. In the LFC network of the grid, the output of each individual system is the phase of its voltage, which is the integration of the angular velocity. The interconnection is power exchanges among the individual systems through transmission lines, which are dependent on phase differences. Thus the LFC network has diffusive output interconnection. Further, individual systems have the phase difference through a transfer function $G_i(s)$, which includes the dynamics of their governor, turbine, generator, and local controller. As shown in our example below, $G_i(s)$ becomes PR when local controllers are suitably designed.

As a second physical motivation, we note that the formation control network of heterogeneous mobile systems can be also represented by (2). In the literature on formation control, either single- or double-integrator networks have been dominantly studied [2]. However, one might construct a formation control network consisting of both of single- and double-integrators. When the integrators are interconnected by relative positions and absolute damping terms are added to double-integrators by local controllers, the heterogeneous network can be represented as (2) with positive real $G_i(s)$.

Actually, a further generalization of (2) is possible and also arises in physical application. In the modeling of the thermal processes in buildings, $G_i(s)$ in (2) is replaced by $G_{ij}(s)$ [5]. However, as we note below, the conditions given to ensure consensus in [5] are quite conservative, leading us to the special case considered in the paper. We further expect that other physical system networks can be represented by (2). Particularly, when individual systems of a network have their own local controllers, one might be able to make $G_i(s)$ PR by suitably designing the local controllers.

In general, network (2) is difficult to analyze if $G_i(s)$ are heterogeneous. If $G_i(s)$ are identical, the network can be decomposed, thereby allowing us to analyze the network readily [6,7]. However, we are not aware of results in the literature that allow us to decompose a network of heterogeneous systems [7]. In [8–10], consensus in a network of heterogeneous linear systems has been studied based on internal model principle. According to [8, 9], it is necessary for heterogeneous systems to embed the internal model for a common trajectory to reach output consensus. In [8], a dynamic interconnection that ensures output consensus has been proposed based on the internal model. The author of [9] has presented a necessary and sufficient condition for output consensus in a network of heterogeneous systems interconnected by relative outputs. Further a dynamic interconnection has been proposed for a network of heterogeneous systems subject to parametric uncertainties [10].

In this paper, we investigate conditions for output consensus in network (2) under the assumption that individual systems might have non-identical dynamics, including different system orders.

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Accordingly, we show that if all $G_i(s)$ in (2) are weakly strictly positive real (WSPR - to be defined below) and the interconnection graph is undirected and connected, network (2) asymptotically reaches output consensus. For the case that all $G_i(s)$ are PR, we also provide a condition for output consensus in network (2), which can be checked by Bode plots of $G_i(s)$. Based on this result, we can analyze a consensus network of output diffusively interconnected linear systems. As an example, we illustrate the results for the LFC problem. Since many local controllers for LFC networks have been designed in the literature and in practice without consideration of stability of the overall network, the conditions provided in this paper might be useful.

Though we focus on a special class of networks addressed in [8–10] in terms of both individual system dynamics and interconnection topology, our result is of interest nonetheless. First, output consensus conditions are given as separate conditions on individual systems. Though a necessary and sufficient condition for output consensus is found in [9], it involves with stability of a matrix that is dependent on the dynamics of every individual system. The dimension of the matrix increases as the number of individual systems increases. It would be difficult to check the condition when the number of individual systems is very large. Our result is useful when constructing a large-scale network because it allows us to interconnect additional individual systems with the existing network without consideration of stability provided that all $G_i(s)$ are WSPR. Second, our result can be applied to any consensus networks of the form (2). In [8,10], dynamic interconnections have been proposed assuming that local outputs are available to individual systems. We emphasize that local outputs are not available to individual systems in some networks, e.g. LFC networks [11]. In the case that such networks are arranged in the form (2), our result can be applied to analysis whereas the dynamic interconnections proposed in [8, 10] cannot be used for them without modifications.

The rest of the paper is organized as follows. In Section 2, we review some required mathematical background. In Section 3, we present conditions for output consensus in a network of PR or WSPR systems cascaded with an integrator. In Section 4.2, we provide some illustrating examples and analyze generator control systems of a LFC network. Concluding remarks are then provided in Section 5.

2. PRELIMINARIES

The set of complex (respectively, real) numbers is denoted by $\mathbb{C}$ (respectively, $\mathbb{R}$). For any $s \in \mathbb{C}$, $\Re[s]$ denotes the real part of s. For any $z \in \mathbb{C}^n$, the conjugate transpose of z is denoted by z^H. The absolute value of z is denoted by $|z|$. For any $A \in \mathbb{C}^{n\times n}$, we denote the spectral norm of A by $\|A\|$. The rank of A is denoted by $\mathrm{Rank}(A)$. The spectrum of A is denoted by $\sigma(A)$. The field values of A, which is defined as set $\{z^H A z \in \mathbb{C} : z \in \mathbb{C}^n, \|z\| = 1\}$, is denoted by $F(A)$. The column vector $[1 \cdots 1]^T \in \mathbb{R}^n$ is denoted by 1_n.

2.1. Graph theory

We consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of nodes and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the set of edges. We assume that there are no self-loops, i.e., for any $i \in \mathcal{V}$, $(i, i) \notin \mathcal{E}$. Each edge $(i, j) \in \mathcal{E}$ has its weight $w_{ij} > 0$. If $(i, j) \in \mathcal{E}$, node j is said to be a neighbor of node i. The set of neighbors of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. A path between two nodes is a sequence of edges such that it is possible to move along the sequence of the edges from one of the nodes to the other. A path does not contain self-loops because we assume that there are no self-loops. If there exists at least one path from any node to any other, the graph is said to be connected. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be undirected if the following conditions hold:

- for any $i, j \in \mathcal{V}$, $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$;
- for any $(i, j) \in \mathcal{E}$ and $(j, i) \in \mathcal{E}$, $w_{ij} = w_{ji}$.

For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let $\mathcal{V} = \{1, \ldots, N\}$. Defining

$$a_{ij} := \begin{cases} w_{ij}, & (i, j) \in \mathcal{E}, \\ 0, & (i, j) \notin \mathcal{E}, \end{cases}$$

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the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined as follows:

$$l_{ij} := \begin{cases} \sum_{k\in\mathcal{N}_i} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

The following is well-known for any Laplacian matrix $L \in \mathbb{R}^{N\times N}$ [12]: (i) $\lambda = 0$ is an eigenvalue of L with its corresponding eigenvector 1_N; (ii) All the non-zero eigenvalues of L are in the open right half of the complex plane. If the graph associated with L is connected, the following is true [12]: (i) $\lambda = 0$ is a distinct eigenvalue of L; (ii) $\lim_{t\to\infty} e^{-Lt} = 1_N v^T$, where v is a vector such that $v^T L = 0$; (iii) There exists a constant c such that the solution of $\dot{x} = -Lx$, where $x = [x_1 \cdots x_N]^T \in \mathbb{R}^N$, satisfies $x_i(t) \to c$ as $t \to \infty$ for all $i = 1, \ldots, N$. Further, if L is the Laplacian matrix of an undirected graph, L is symmetric and positive semi-definite.

2.2. *Positive real systems*

Let $h(s)$ be a rational transfer function. That is, $h(s)$ is a function of a complex variable s that can be written as the ratio of two polynomial functions whose coefficients are real. Positive realness of $h(s)$ is defined as follows:

Definition 2.1
[13] A proper rational transfer function $h(s)$ is said to be positive real if

(i) $h(s)$ has no poles in $\Re[s] > 0$;
(ii) $h(s)$ is real for positive real s;
(iii) $\Re[h(s)] \geq 0$ for all $\Re[s] > 0$.

The following theorem provides a necessary and sufficient condition for the positive realness.

Theorem 2.1
[13] A proper rational transfer function $h(s)$ is positive real if and only if

(i) $h(s)$ has no poles in $\Re[s] > 0$;
(ii) $\Re[h(j\omega)] \geq 0$ for all ω such that $j\omega$ is not a pole in $h(s)$;
(iii) If $s = j\omega_0$ is a pole in $h(s)$, it is a simple pole. If ω_0 is finite, the residual $\lim_{s\to j\omega_0}(s - j\omega_0)h(s)$ is real and positive. If ω_0 is infinite, the limit $\lim_{\omega\to\infty} h(j\omega)/j\omega$ is real and positive.

If there exists $\epsilon > 0$ such that $h(s - \epsilon)$ is PR, then $h(s)$ is said to be strictly positive real (SPR). The following property of weakly strictly positive realness is stronger than positive realness while weaker than strict positive realness [13]:

Definition 2.2
[13] A proper rational transfer function $h(s)$ is said to be weakly strictly positive real if

(i) $h(s)$ is has no poles in $\Re[s] \geq 0$;
(ii) $\Re[h(j\omega)] > 0$ for all $\omega \in (-\infty, \infty)$.

From *Definition 2.2*, a WSPR function $h(s)$ satisfies conditions (i) and (ii) in *Theorem 2.1*. Further, $h(s)$ automatically satisfies condition (iii) in *Theorem 2.1* because it does not have a pole on the imaginary axis. Thus we see that a WSPR function is PR.

3. MAIN RESULT

We consider the following N dynamical systems described in s-domain:

$$Y_i(s) = \frac{1}{s} G_i(s) U_i(s),\ i = 1, \ldots, N, \tag{3}$$

where $Y_i(s) \in \mathbb{R}$ are the outputs, $U_i(s) \in \mathbb{R}$ are the interconnection inputs defined below, and

$$G_i(s) = \frac{N_i(s)}{D_i(s)},\ D_i(s) := \sum_{k=0}^{n_i-1} d_{ik}s^k,\ N_i(s) := \sum_{k=0}^{m_i} n_{ik}s^k,\ i = 1, \ldots, N.$$

Without loss of generality, let $d_{i(n_i-1)} = 1$ for all $i = 1, \ldots, N$. To avoid the special case where all nodes are stable, we assume that $n_{i0} \neq 0$ for all $i = 1, \ldots, N$, which implies that every node has a pole at zero. Then the degree of the denominator of $Y_i(s)/N_i(s)$ is given by n_i for all $i = 1, \ldots, N$. Assuming that interconnections among individual systems (3) are modeled by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we consider interconnection inputs $U_i(s)$ given as

$$U_i(s) = -\sum_{j \in \mathcal{N}_i} a_{ij} \left(Y_i(s) - Y_j(s)\right),\ i = 1, \ldots, N. \tag{4}$$

The network of individual systems (3) under interconnection inputs (4) is described as

$$sY_i(s) = -\sum_{j \in \mathcal{N}_i} a_{ij} G_i(s) \left(Y_i(s) - Y_j(s)\right),\ i = 1, \ldots, N. \tag{5}$$

Let $Y(s) := [Y_1(s) \cdots Y_N(s)]^T$ and $G(s) := \mathrm{diag}\left(G_1(s), \ldots, G_N(s)\right)$. Network (5) can be arranged as

$$sY(s) = -G(s)LY(s), \tag{6}$$

where L is the Laplacian matrix of $\mathcal{G}$. Let $y_i(t)$ be the inverse Laplace transform of $Y_i(s)$ for all $i = 1, \ldots, N$. We say that network (5) asymptotically reaches output consensus if $y_i(t) - y_j(t) \to 0$ as $t \to \infty$ for all $i = 1, \ldots, N$.

Several comments are in order here. First, we are interested in investigating separate conditions on individual systems (3) that ensure the output consensus in network (5) under the assumption that $\mathcal{G}$ is connected. According to [14], network (5) asymptotically reaches output consensus if and only if

- (*graph connectedness*) The graph $\mathcal{G}$ is connected;
- (*pole location*) The following characteristic equation of network (6),

$$\det(sI_N + G(s)L) = 0, \tag{7}$$

  has a distinct root at zero and all the other roots are in the open left half of the complex plane.

It would be difficult to solve (7) if the number of the individual systems is very large. As opposed to solving (7) directly, we seek to investigate properties of the individual systems that ensure output consensus assuming that $\mathcal{G}$ is connected. A necessary and sufficient condition for output consensus is found in [9, *Theorem 3*]. Though the condition can be used for analysis of network (5), it involves with stability of a matrix that is dependent on all $G_i(s)$ in general. It would be difficult to check the condition if N is very large.

Second, if $G_i(s)$ are identical, (7) can be decomposed into low-order equations [6, 7], which are simpler to analyze than (7). To see this, suppose that $G_i(s) = G_1(s)$ are identical, (7) is reduced into $\det(sI_N + G_1(s)L) = 0$. Since we consider the undirected case, L is symmetric, there exists an orthogonal matrix U such that $L = U\Lambda U^T$, where Λ is a diagonal matrix whose diagonal elements are the eigenvalues of L. Thus we have

$$\begin{aligned}
\det(sI_N + G_1(s)L) &= \det(sI_N + G_1(s)U\Lambda U^T) \\
&= \det(U)\det(U^T)\det(sI_N + G_1(s)\Lambda) \\
&= \prod_{i=1}^{N}(s + \lambda_i G_1(s)),
\end{aligned}$$

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where $0 = \lambda_1 \leq \cdots \leq \lambda_N$ are the eigenvalues of L. To ensure that (7) has a distinct root at zero and all the other roots at the open left half of the complex plane (pole location condition), it is required that the root locus of $(1 + kG_1(s)/s)$ lies in the open left half of the complex plane for all $\lambda_2 \leq k \leq \lambda_N$. However, as mentioned in [7], it is not certain whether (7) can be decomposed when $G_i(s)$ are not identical. Thus we attempt to analyze the roots of (7) based on properties of the field values of $G(s)$ and L assuming that $G_i(s)$ are PR or WSPR. This allows us to analyze locations of the roots in the complex plane without solving (7).

Third, network (5) has the same form as the dynamic consensus networks found in [5, 14]. The authors of [14] have studied a consensus network of single integrators whose interconnection channels are modeled by stable linear time-invariant systems. Motivated by thermal processes in buildings, the authors of [5] have studied room temperature dynamics in a building, which is modeled as a consensus network of single integrators whose interconnection coefficients are modeled by rational functions. Both consensus networks found in [5, 14] are single integrator networks whose interconnection coefficients are given by stable transfer functions. Specifically, the consensus networks are described as $sY(s) = -\Gamma(s)Y(s)$, where $\Gamma(s)$ is the dynamic Laplacian matrix[†] of an interconnection graph whose edges are given by transfer functions. According to the results in [5, 14], $sY(s) = -\Gamma(s)Y(s)$ asymptotically reaches output consensus if $\Gamma(0)$ is a static Laplacian matrix of a connected graph and $\Gamma(s) = [\gamma_{ij}(s)]$ satisfies the following diagonal dominance condition:

$$\Re\left[\gamma_{ii}(s)\right] > \sum_{j\neq i, j=1}^{N} |\gamma_{ij}(s)|, \ \Re[s] \geq 0, s \neq 0. \tag{8}$$

However, network (5) does not satisfy (8) because $G(s)L$ (or $\Gamma(s)$ in this case) satisfies the following equalities:

$$l_{ii}G_i(s) = \sum_{j\neq i, j=1}^{N} l_{ij}G_i(s), \ i = 1, \ldots, N, \tag{9}$$

which fail to provide us with an useful estimate for locations of the roots of (7). To clarify this, we apply Gershgorin circle theorem [15] to $-G(s)L$. Let $\lambda(s)$ be an eigenvalue of $-G(s)L$ and $v(s) = [v_1(s) \cdots v_n(s)]^T$ be a corresponding eigenvector. Let $k \in \{1, \ldots, N\}$ be chosen so that $|v_k(s)| = \max_{1\leq i\leq N} |v_i(s)|$. From the fact that $-G(s)Lv(s) = \lambda(s)v(s)$, we get

$$|\lambda(s) + l_{kk}G_k(s)| = \left| \frac{\sum_{j\neq k, j=1}^{N} l_{kj}G_k(s)v_j(s)}{v_k(s)} \right| \leq \sum_{j\neq k, j=1}^{N} l_{kj}\left|G_k(s)\right| = l_{kk}\left|G_k(s)\right|, \tag{10}$$

where we use (9). We see that (10) does not ensure that $\Re[\lambda(s)] \geq 0$ unless $G_k(s)$ is real-valued.

With the aforementioned reasons, we analyze the root locations of (7) in the complex plane based on properties of the field values of $G(s)$ and L to investigate properties of the individual systems that ensure output consensus in network (5) assuming that $\mathcal{G}$ is connected. The following theorem is useful:

Theorem 3.1
[16] Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. If B is symmetric and positive semi-definite, then, for any $\lambda \in \sigma(AB)$, there exist $a \in F(A)$ and $b \in F(B)$ such that $\lambda = ab$.

Then the following theorem provides a sufficient condition for output consensus in network (6):

Theorem 3.2
For network (5), assume that

[†]Since the edges of the interconnection graph are dynamical systems, the elements of the Laplacian matrix are functions of s in s-domain. Thus we call the Laplacian matrix dynamic.

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(a) The interconnection graph associated with L is connected;
(b) $G_i(s)$ are WSPR for all $i = 1, \dots, N$.

Then,

(i) $s = 0$ is a distinct (not repeated) root of (7);
(ii) All the non-zero roots of (7) are in the open left half of the complex plane;
(iii) For any initial condition, there exists a constant c such that $y_i(t) \to c$ as $t \to \infty$ for all $i = 1, \dots, N$.

Proof
(i) First, $\mathrm{Rank}(L) = N - 1$ from condition (a). From condition (b), we have $G_i(0) > 0$ for all $i \in 1, \dots, N$, which implies that $G(0)$ is full-rank. It follows that $\mathrm{Rank}(G(0)L) = \mathrm{Rank}(L) = N - 1$. Hence $s = 0$ is a distinct (not repeated) root of (7).

(ii) We first show that all the non-zero roots of (7) have non-positive real-parts. Suppose that s^* is a root of (7) with positive real-part, i.e., $\Re[s^*] > 0$. Since L is symmetric and positive semi-definite, it follows from *Theorem 3.1* that there exist $a \in F\left(-G(s^*)\right)$ and $b \in F(L)$ such that $s^* = ab \in F(-G(s^*)L)$. Note that, for any $z = [z_1 \cdots z_N]^T \in \mathbb{C}^N$,

$$z^H G(s^*) z = \sum_{i=1}^{N} z_i^H G_i(s^*) z_i = \sum_{i=1}^{N} G_i(s^*)|z_i|^2.$$

From condition (b), we have $\Re[G_i(s^*)] \geq 0$ for all $i \in 1, \dots, N$ because $\Re[s^*] > 0$. Then it follows that $\Re\left[-z^H G(s^*) z\right] = -\sum_{i=1}^{N} \Re[G_i(s^*)]|z_i|^2 \leq 0$, which implies that $\Re[a] \leq 0$. Further, we have $b \geq 0$ from the symmetry and the positive semi-definiteness of L. Thus $\Re[s^*] = \Re[ab] \leq 0$, which is a contradiction. This implies that all the non-zero roots of (7) have non-positive real-parts.

To complete the proof, we next show that (7) does not have pure imaginary roots. Suppose that $j\omega^*$ is a pure imaginary root of (7), which means that $j\omega^* \in F(-G(j\omega^*)L)$. Then, there exist $a \in F\left(-G(j\omega^*)\right)$ and $b \in F(L)$ such that $j\omega^* = ab$. From condition (b), we have $\Re[G_i(j\omega^*)]$ for all $i = 1, \dots, N$. Hence, for any $z \in \mathbb{C}^N$, $\Re\left[-z^H G(j\omega^*) z\right] = -\sum_{i=1}^{N} \Re[G_i(j\omega^*)]|z_i|^2 < 0$, which implies that $\Re[a] < 0$. Further, $b \neq 0$ because $j\omega^*$ is pure imaginary. Thus we have $\Re[j\omega^*] = \Re[ab] < 0$, which is a contradiction. Therefore there exist no pure imaginary roots of (7), which completes the proof.

(iii) This claim can be proved based on a similar argument in the proof of *Theorem 4.1* in [5]. Since $s = 0$ is a distinct root of (7), this claim is proved by showing that no solutions of (6) exist with modes corresponding to values in $\mathbb{S} := \{s \in \mathbb{C} : \Re(s) \geq 0, s \neq 0\}$. Let $G(s) = N(s)/D(s)$. According to [17], the solutions of (6) are given by $w(t)$ that satisfy

$$\left(sD(s) + N(s)L\right)\big|_{s=\frac{d}{dt}} w(t) = 0.$$

Further, it is known that all the allowable modes of $w(t)$ are given by the roots of $\det(sD(s) + N(s)L) = 0$. Since $sD(s)$ has no zeros in $\mathbb{S}$ due to the weakly strictly positive realness condition of $G_i(s)$, the roots of $\det(sD(s) + N(s)L) = 0$ in $\mathbb{S}$ are identical to those of (7). Since there are no roots of (7) in $\mathbb{S}$, it follows that no solutions of (6) have modes corresponding to values in $\mathbb{S}$. Therefore, for any initial condition, there exists a constant c such that $y_i(t) \to c$ as $t \to \infty$. $\square$

Remark 3.1
We provide some remarks on *Theorem 3.2*:

- *Theorem 3.2* is particularly useful for constructing a large-scale network because it allows one to interconnect additional individual systems with the existing network without consideration of stability of the resulting network provided that all $G_i(s)$ are WSPR and the interconnection graph is connected.
- As shown in (10), an application of Gershgorin circle theorem [15] to $-G(s)L$ by using (9) fails to provide us with a useful bound for locations of the roots of (7).

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- Regarding the proof of the statement that (7) has no pure imaginary roots, it should be noted that, for $A, B \in \mathbb{C}^{N\times N}$, $\sigma(AB) \subseteq \sigma(A)\sigma(B)$ fails to hold [15]. That is, positive semi-definiteness of L together with the fact that $G(jw)$ does not have a pure imaginary eigenvalue for all $\omega \in (-\infty, \infty)$ does not imply that there is no purely imaginary root of (7).

Next we consider the case that all $G_i(s)$ are PR with no poles on the imaginary axis. Different from the previous WSPR case, we need to consider two things. First, in the proof of *Theorem 3.2*, we need the condition that $\Re[G(j\omega)] > 0$ for all $\omega \in (-\infty, \infty)$ to show that (7) does not have pure imaginary roots. Note that we have only $\Re[G_i(j\omega)] \geq 0$ in this case, which is based on condition (ii) in *Theorem 2.1*. That is, there might be pure imaginary roots of (7). Second, to ensure that $s = 0$ is a distinct root of (7), we need an additional condition that $G_i(0) > 0$ for all $i = 1, \dots, N$. Because of this additional condition, the small gain theorem fails to provide a condition for output consensus in the network in general. According to the small gain theorem, if $\| - G(j\omega)L/j\omega\| < 1$ for all $\omega \in (-\infty, \infty)$, output consensus in network (5) is ensured. However, note that $\lim_{\omega\to 0} \| - G(j\omega)L/j\omega\| = \infty$ when $G_i(0) > 0$ for all $i = 1, \dots, N$.

In frequency responses of many physical systems, phases are bounded in low frequency whereas magnitudes are bounded in high frequency. Based on this observation, we suppose that there exists a constant $\omega_s > 0$ such that $\Re[G_i(j\omega)] > 0$ for all $0 < \omega \leq \omega_s$ and $\max_{\omega\geq\omega_s} \|G(j\omega)\|$ is bounded. Such ω_s can be found from the bode plots of $G_i(j\omega)$. To show that (7) does not have pure imaginary roots, we then attempt to utilize the condition that $\Re[G(j\omega)] > 0$ for $0 < \omega \leq \omega_s$ and apply the small gain theorem for $\omega \geq \omega_s$. Based on this idea, we present the following theorem:

Theorem 3.3
For network (5), assume that

(a) The graph associated with L is connected;
(b) $G_i(s)$ are PR and have no poles on the imaginary axis for all $i = 1, \dots, N$;
(c) $G_i(0)$ are positive, i.e., $n_{i0}/d_{i0} > 0$, for all $i = 1, \dots, N$;
(d) There exists $\omega_s > 0$ such that $\Re[G_i(j\omega)] > 0$ for all $0 < \omega \leq \omega_s$ and

$$\max_{i\in\{1,\dots,N\}} \max_{\omega\geq\omega_s} |G_i(j\omega)| < \frac{\omega_s}{\lambda_{max}(L)},$$

where $\lambda_{max}(L)$ is the largest eigenvalue of L.

Then the statements in *Theorem 3.2* are true.

Proof
(i) From condition (c), $G(0)$ is positive definite. Then it follows from condition (a) that $s = 0$ is a distinct root of (7).

(ii) From condition (b), we have $\Re[G_i(s)] \geq 0$ for all $\Re[s] > 0$. Then it can be shown that there are no non-zero roots of (7) in the open right half of the complex plane in the same way as the proof of *Theorem 3.2*.

The proof of this claim is completed by showing that (7) does not have pure imaginary roots. First, suppose that there exists a non-zero root $j\omega^*$ of (7) such that $|\omega^*| \leq \omega_s$. Based on condition (d), we can show that $j\omega^*$ cannot be a root of (7) in the same way as the proof of *Theorem 3.2*.

Second, suppose that there exists a non-zero root $j\omega^*$ of (7) such that $|\omega^*| \geq \omega_s$. Then we have

$$\det(j\omega^* I_N + G(j\omega^*)L) = \det\left(I_N + \frac{G(j\omega^*)L}{j\omega^*}\right) = 0.$$

Due to condition (d), we have

$$\left\|\frac{G(j\omega^*)L}{j\omega^*}\right\| \leq \frac{\lambda_{max}(L)\max_{i\in\{1,\dots,N\}}\max_{\omega\geq\omega_s}|G_i(j\omega)|}{\omega_s} < 1,$$

which is a contradiction, because of the small gain theorem. Therefore there are no non-zero pure imaginary roots of (7).

(iii) The proof of this claim is the same as the proof of *Theorem 3.2*. $\square$

4. EXAMPLES

4.1. Illustrating examples

Consider the following network consisting of N_s single and N_d $(= N - N_s)$ double integrators over an undirected graph:

$$\dot{y}_i = u_i,\ i = 1, \ldots, N_s \tag{11a}$$
$$\ddot{y}_i + k_{vi}\dot{y}_i = u_i,\ i = N_s + 1, \ldots, N, \tag{11b}$$

where $y_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$. In (11b), the absolute damping terms $k_{vi}\dot{y}_i$ are due to the local controllers of the double-integrators. The interconnection inputs are given as

$$u_i = -k_{pi} \sum_{j \in \mathcal{N}_i} a_{ij} \left(y_i - y_j\right),\ i = 1, \ldots, N.$$

Then network (11) can be represented as (6) with

$$G(s) = -\operatorname{diag}\left(k_{p1}, \ldots, k_{pN_s}, \frac{k_{p(N_s+1)}}{s + k_{v(N_s+1)}}, \ldots, \frac{k_{pN}}{s + k_{vN}}\right),$$

which satisfies condition (b) in *Theorem 3.2* under the assumption that all the absolute damping terms $k_{vi}\dot{y}_i$ have positive coefficient, i.e., $k_{vi} > 0$. Thus it can be concluded that network (11) asymptotically reaches output consensus if the interconnection graph is connected.

Consider the following network over an undirected graph:

$$\dot{x}_i = A_i x_i + B_i \sum_{j \in \mathcal{N}_i} a_{ij} \left(y_j - y_i\right), \tag{12a}$$
$$y_i = C_i x_i, \tag{12b}$$

where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times 1}$, and $C_i \in \mathbb{R}^{1 \times n_i}$. Suppose that A_i has a distinct zero eigenvalue and further let $C_i(sI_{n_i} - A_i)^{-1}B_i = G_i(s)/s$. Then the network can be arranged in the form of (5). In this case, if the graph is connected and $G_i(s)$ satisfy the conditions of *Theorem 3.2* or *3.3*, network (12) asymptotically reaches output consensus.

Recall that condition (ii) in *Theorem 2.1* implies that if a proper rational transfer function is positive real, its relative degree is at most one. For network (5), let $G_i(s)$ be positive real for all $i = 1, \ldots, N$. Then network (5) can be described as (12) with

$$A_i = \begin{bmatrix} -d_{i(n_i-1)} & 1 & 0 & \cdots & 0 \\ -d_{i(n_i-2)} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_{i0} & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},\ B_i = [n_{in_i} \cdots n_{i0}]^T,\ C_i = [1\ 0 \cdots 0], \tag{13}$$

where $n_{in_i} = 0$ if the relative degree of $G_i(s)$ is one. According to [8,9], heterogeneous systems are able to reach consensus only when each of them contains the internal reference model for a common trajectory. For network (12), a necessary condition for all $y_i(t)$ to approach a common value is that, for any $c \in \mathbb{R}$, there exist $x_{i0} \in \mathbb{R}^{n_i}$ such that $C_i e^{A_i t} x_{i0} = c$ for all $i = 1, \ldots, N$ [9, Theorem 1]. When A_i, B_i, and C_i are given as (13), the condition is always satisfied, i.e., for any $c \in \mathbb{R}$, $x_{i0} = c[1\ d_{i(n_i-1)} \cdots d_{i0}]^T$ satisfies $C_i e^{A_i t} x_{i0} = c$. According to [8, *Theorem 3*], if all $y_i(t) \to c$ for some $c \in \mathbb{R}$, there exist $S \in \mathbb{R}$, $R \in \mathbb{R}$, and $\Pi_i \in \mathbb{R}^{n_i \times 1}$ such that $A_i\Pi_i = \Pi_i S,\ C_i\Pi_i = R$ for all $i = 1, \ldots, N$. Let $S = 0$ and $R = 1$. Then $\Pi_i = [1\ d_{i(n_i-1)} \cdots d_{i0}]^T$ satisfies $A_i\Pi_i = \Pi_i S,\ C_i\Pi_i = R$.

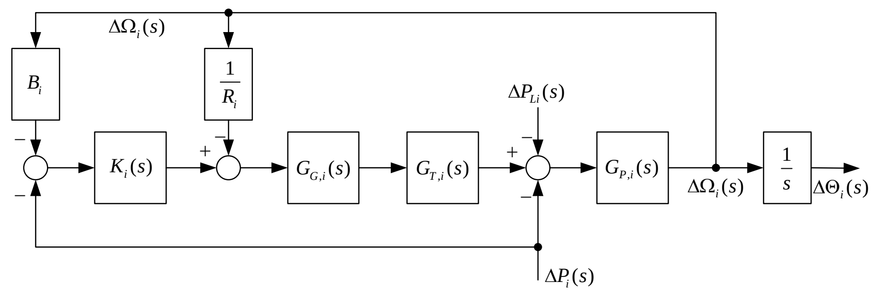

Figure 2. Individual system model in an LFC network.

4.2. Load frequency control network of synchronous generators

A synchronous generator system of an LFC network can be represented as the block diagram illustrated in Figure 2 [11]. In Fig. 2, $K_i(s)$, $G_{Gi}(s)$, $G_{Ti}(s)$, and $G_{Pi}(s)$ denote the local controllers, governors, turbines, and generators, and $\Delta\Theta_i(s)$ and $\Delta\Omega_i(s)$ denote the phase and frequency variations of the voltages of the individual systems. The local load and the power transfer variations are denoted by ΔP_{Li} and $\Delta P_i(s)$. The power transfer variations are dependent on the phase differences [11]:

$$\Delta P_i(s) = \sum_{j\in\mathcal{N}_i} T_{ij}\left(\Delta\Theta_i(s) - \Delta\Theta_j(s)\right),$$

where $T_{ij} = T_{ji}$ are the synchronizing power coefficients. Since $T_{ij} = T_{ji}$, the interconnection of the LFC network can be modeled by an undirected graph.

Consider the following LFC network, whose interconnection graph is $\mathcal{G}$. From Figure 2, we obtain

$$s\Delta\Theta_i(s) = G_i(s)\sum_{j\in\mathcal{N}_i} T_{ij}\left(\Delta\Theta_j(s) - \Delta\Theta_i(s)\right) - G_{Li}(s)\Delta P_{Li}(s),\ i = 1,\ldots,N, \tag{14}$$

where $G_i(s)$ and $G_{Li}(s)$ are given as

$$G_i(s) = \frac{G_{P,i}(s)\left[1 + K_i(s)G_{G,i}(s)G_{T,i}(s)G_{P.i}(s)\right]}{1 + \left[1/R_i + B_iK_i(s)\right]G_{G,i}(s)G_{T,i}(s)G_{P,i}(s)},$$
$$G_{Li}(s) = \frac{G_{P,i}(s)}{1 + \left[1/R_i + B_iK_i(s)\right]G_{G,i}(s)G_{T,i}(s)G_{P,i}(s)}.$$

The objective of LFC network (14) is regulating $\Delta\omega_i$ and ΔP_i by appropriately designing $K_i(s)$. Note that, if there exists a constant c such that $\Delta\theta_i \to c$ for all $i = 1,\ldots,N$, $\Delta\omega_i \to 0$ and $\Delta P_i \to 0$. Conversely, if $\Delta\omega = 0$ and $\Delta P_i = 0$, $\Delta\theta_i$ have a common constant value. In other words, the objective of LFC network (14) is to achieve output consensus.

Assuming that $\mathcal{G}$ is connected and $\Delta P_{Li}(s) \equiv 0$, we can check whether network (14) reaches output consensus by checking if $G_i(s)$ are WSPR. Figure 3 shows the Nyquist plots for $G_i(s)$ of a typical steam turbine generator and a hydraulic generator systems, whose parameters are given in [11, pp. 598–599]. For both generators, $K_i(s) = 0.01/s$ is adopted as suggested in [18]. From Figure 3, we can see that the Nyquist plots do not cross the imaginary axes. Since $G_i(s)$ do not have poles in the closed right-half of the complex plane, they are WSPR. Note that $G_{Li}(s)$ have a zero at $s = 0$ in both cases. This implies that constant local load variations can be rejected.

Many local controllers in LFC networks proposed in existing results have been designed without consideration of stability of the networks. It would be desirable if local controllers in LFC networks are designed considering *Theorems 3.2* or *3.3*.

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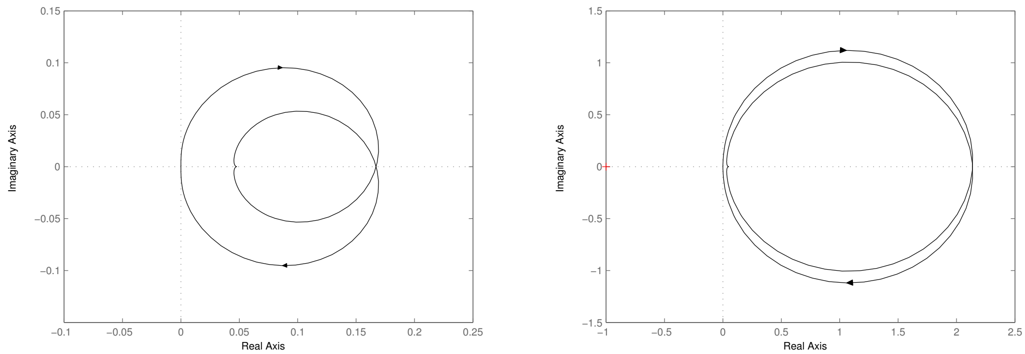

(a) Steam turbine generator case. (b) Hydraulic generator case.

Figure 3. Nyquist plots for $G_i(s)$ of synchronous generators with typical parameters.

5. CONCLUSION

For a network of PR or WSPR systems cascaded with a single integrator, we provided sufficient conditions for achieving output consensus. We showed that an LFC network of synchronous generator systems could be analyzed based on these results.

There are several possible research directions. First, we considered only undirected networks. One may next consider the directed networks. Second, the results in this paper do not cover networks of oscillatory or unstable systems. It would be interesting to study such networks. Third, we addressed only single-input/single-output system networks. One may extend the results in this paper to multi-input/multi-output networks.

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