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Dynamic Consensus Networks with Application to the Analysis of Building Thermal Processes ⋆

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Abstract: We consider consensus networks whose nodes are integrators and whose edges are 2-tuples of real rational functions representing dynamical systems that couple the nodes. We review salient points from graph theory, including Laplacians, interconnection matrices, and consensus protocols, all of which typically involve constructs with static weights. We then generalize these notions to the case of graphs with integrating nodes and dynamic edges. We give conditions under which such graphs admit consensus, meaning that in the steady-state the node variables converge to a common value. The ideas are illustrated an example that motivated this work: the modeling of thermal processes in buildings.

Keywords: Consensus networks, dynamic graphs, building efficiency, thermal models.

1. INTRODUCTION

Recently there has been considerable interest in what is called the consensus problem. In this problem we suppose N agents n_i evolve their individual belief ξ_i about a global consensus variable ξ using nearest neighbor communications according to the consensus protocol

$$\dot{\xi}_i = -\sum_{j\in\mathcal{N}_i} \lambda_{ij}(\xi_i - \xi_j), \tag{1}$$

where $\mathcal{N}_i$ is the collection of indices of all agents n_j with whom agent n_i can communicate. Readers can refer to Olfati-Saber and Murray (2007) and the references therein for a detailed review of this problem. Here we simply note that to our best knowledge most studies in the literature have focused on the case where the weights λ_{ij} are constants (or random variables). Motivated by the problem of modeling the thermal processes in buildings, in this paper we present a generalization of the consensus problem whereby the weights are no longer static gains, but instead represent dynamical systems.

Specifically, we consider problems of the form

$$\Xi_i(s) = -\frac{1}{s}\sum_{j\in\mathcal{N}_i}\left[\lambda^S_{ij}(s)\Xi_i(s) - \lambda^C_{ij}(s)\Xi_j(s)\right], \tag{2}$$

where $\Xi_i(s)$ is the Laplace transform of $\xi_i(t)$ and we define the real rational (transfer) functions $\lambda^S_{ij}(s)$ and $\lambda^C_{ij}(s)$ to be the self-correction term and the cross-correction term, respectively, for the node n_i. With this notation, we specifically mean that the consensus variable ξ_i is calculated via

$$\dot{\xi}_i = -\sum_{j\in\mathcal{N}_i} q^S_{ij} - q^C_{ij} \tag{3}$$

where q^S_{ij} is calculated via the differential equation

$$B^S_{ij}\left(\frac{d}{dt}\right)q^S_{ij} = A^S_{ij}\left(\frac{d}{dt}\right)\xi_i, \tag{4}$$

where $A^S_{ij}(s)$ and $B^S_{ii}(s)$ are the numerator and denominator of $\lambda^S_{ij}(s)$. q^C_{ij} is calculated similarly.

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We will refer to (1) as the *static consensus* protocol and (2) or (3) as the *dynamic consensus* protocol. Note then that the dynamic consensus protocol extends the standard static consensus protocol in two ways: (1) the connection variables are transfer functions, and (2) we allow different connection weights to multiply the current estimate of the node and the estimates of other nodes.

Many systems fall into the dynamic consensus framework, particularly large scale systems described by interconnected storage elements. A specific example of this that will be illustrated here is thermal processes in buildings. When two rooms are separated by a wall (see Fig. 1) and there are no external heat flows, the temperatures in each room can be described by

$$T_i(s) = -\frac{1}{C^r_i s}\left[\frac{A_{ij}(s)}{B_{ij}(s)}T_i(s) - \frac{D_{ij}(s)}{B_{ij}(s)}T_j(s)\right], \tag{5}$$

where the various transfer functions are defined later in the paper (following Lee and Braun (2008)). Similar to (2) we see a cross-correction term and a self-correction term and, when applying (5) to several interconnected rooms in a building, the resulting expression can be seen to have the same form as (2).

While the consensus behavior of uncontrolled building thermal dynamics is fairly intuitive, the dynamic consensus framework proposed here also provides a framework for any control implementation that results in modification of link weights. Our goal is to develop performance criterion that clearly exhibit the effect of system structure (i.e. interconnections) and can be checked using information "local" to each node.

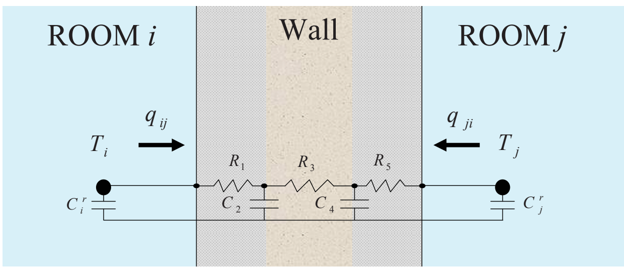

Fig. 1. Two rooms connected by a wall using the 3R2C model.

The paper is organized as follows. We begin with a review of some essential facts about graphs, interconnection matrices, and consensus. We then formulate the idea of dynamic consensus by defining a class of directed graphs characterized by nodes that are integrators and arcs described by real-rational transfer functions. For such graphs we can define what we call the dynamic Laplacian of the system. We then present our main result: when the "DC-gain" of the dynamic Laplacian is an interconnection matrix and a particular diagonal dominance condition is satisfied by the weights $\lambda_{ij}^S(s)$ and $\lambda_{ij}^C(s)$, the graph admits consensus, meaning all node variables converge to the same value. These results are then illustrated with an example of modeling the thermal processes in a building. The paper concludes with comments on future research on this topic.

2. GRAPHS, INTERCONNECTION MATRICES, AND CONSENSUS

Here we collect key points about graphs, interconnection matrices, and consensus problems. While much of this section is standard, in fact we have found a fair amount of confusion in the consensus literature related to definitions of graph-theoretic concepts. Unfortunately, space limitations preclude clarifying these concepts here. Thus we simply define the terms the way they are most commonly used. Most of our notation and results follow Tuna (2009); Egerstedt and Ji (2007); Caughman and Veerman (2006); and Ren et al. (2003).

2.1 Graphs

Consider a weighted, directed graph as shown in Fig. 2. Such graphs can be described as a set of nodes (or vertices) $\mathcal{N} = \{n_i\}$ connected by a set of arcs $\mathcal{E} = \{(n_i, n_j) : n_i, n_j \in \mathcal{N}\}$ (called edges if the graph is undirected). We assume there are no self-loops associated with any node. Following typical convention for directed graphs, we associate the first entry in the ordered pair (n_i, n_j) with the "tail" of the arc and the second with the "head" of the arc. If there is an edge between nodes n_i and n_j we say these nodes are adjacent (or neighbors). We denote the *out-degree* neighbors of node n_i as $\mathcal{N}_i = \{j : (n_i, n_j) \in \mathcal{E}\}$; that is, the set of nodes n_j adjacent to n_i that have a head of an arc touching them that originates from node n_i. Each arc has an associated weight λ_{ij} [1]. A path between two nodes is a sequence of arcs by which it is possible to "move" along the sequence of arcs from one of the nodes to the other. If there is at least one node that has at least one path to every other node, the graph is said to be connected.

The (weighted) Laplacian matrix of a (weighted) graph is an important matrix in graph theory and its spectral properties can be used to infer many facts about the graph. When a graph is undirected there is no ambiguity in the definition of the Laplacian, which is constructed using what are called degree and adjacency matrices. However, in the case of directed graphs the definitions found in the literature can be confusing as the Laplacian can be defined in in terms of the in– or out-degree and in– and out-adjacency matrices (defined according to the number of arcs entering or leaving a node, respectively), using either an in-degree or out-degree convention for defining neighbors, resulting in four possible ways to define the Laplacian. Again, in this paper we adopt an *out-degree* convention using the labeling scheme as shown in Fig. 2, where a weight λ_{ij} is interpreted as the weight associated with an arc that leaves node n_i and goes to node n_j. Thus we define the associated graph Laplacian $L = [l_{ij}]$ to be

$$l_{ij} = \begin{cases} \sum_{k \in \mathcal{N}_i} \lambda_{ik} & i = j, \\ -\lambda_{ij} & i \neq j \text{ and } (i,j) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

To illustrate, for the (weighted) graph shown in Fig. 2 the associated (weighted) Laplacian is given by

$$\begin{bmatrix} \lambda_{12}+\lambda_{13} & -\lambda_{12} & -\lambda_{13} & 0 & 0 & 0 \\ -\lambda_{21} & \lambda_{21}+\lambda_{24} & 0 & -\lambda_{24} & 0 & 0 \\ 0 & 0 & \lambda_{35} & 0 & -\lambda_{35} & 0 \\ 0 & 0 & 0 & \lambda_{45} & -\lambda_{45} & 0 \\ 0 & -\lambda_{52} & -\lambda_{53} & 0 & \lambda_{52}+\lambda_{53}+\lambda_{56} & -\lambda_{56} \\ 0 & 0 & 0 & -\lambda_{64} & 0 & \lambda_{64} \end{bmatrix}.$$

A key result from graph theory is that a graph is connected if and only if $\lambda = 0$ is a distinct eigenvalue of the Laplacian matrix (Chung (1997)).

2.2 Interconnection Matrices

A matrix $\Lambda = [\lambda_{ij}]$ is said to be an interconnection (or an interconnection matrix) if its elements satisfy $\lambda_{ij} \geq 0$ for $i \neq j$ and

$$\lambda_{ii} = -\sum_{i \neq j} \lambda_{ij}.$$

Notice that the Laplacian defined above is an interconnection matrix multiplied by -1. Thus it is common to talk about the graph associated with (or induced by) a given interconnection matrix and to view the interconnection matrix of such as graph as the negative of the graph's Laplacian. We will say the interconnection Λ is connected if its associated graph is connected.

For *any interconnection* matrix the following are true (these are standard results from graph theory):

(1) The row sums of Λ are all zero.
(2) $\lambda = 0$ is an eigenvalue of Λ with eigenvector $\mathbf{1}$.
(3) The matrix is diagonally dominant.
(4) All non-zero eigenvalues are in the open left-half of the complex plane.

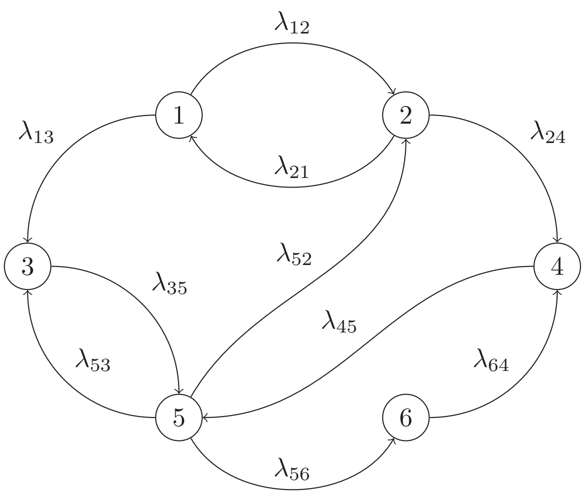

Fig. 2. Weighted directed graph.

[1] We comment that in the consensus literature it is more common to define the neighbors of a node using an *in-degree* convention. In an in-degree convention, neighbors of node n_i are defined to be $\mathcal{N}_i = \{j : (n_j, n_i) \in \mathcal{E}\}$; that is, the set of nodes adjacent to n_i that have a tail of an arc touching them that ends at n_i. In this case the weight associated with arc (n_j, n_i) is typically defined to be λ_{ij}. Because we are motivated by a physical model that defines dynamics in terms of flows (of energy) leaving a node, here we use an out-degree convention. However, there is no loss of generality or consistency in our approach.

(5) The matrix exponential $e^{\Lambda t}$ has row sum equal one for all t (i.e., is a stochastic matrix).

For a *connected interconnection matrix* the following are also true:

(1) $\lambda = 0$ is a distinct eigenvalue of Λ.
(2) $e^{\Lambda t} \to \mathbf{1}v^T$ where v is a vector such that $v^T\Lambda = 0$ (i.e., the left eigenvector of the eigenvector $\Lambda = 0$) and $\sum v_i = 1$.
(3) For a vector $x = [x_1, x_2, \ldots, x_N]^T$, the solution of $\dot{x} = \Lambda x$ satisfies $x_i \to x^*$ for some constant x^* (i.e., consensus - see below).

2.3 Consensus Networks

As noted above, the multi-agent consensus problem supposes that N agents evolve their individual belief ξ_i about a so-called global consensus variable ξ using nearest neighbor communications according to the consensus protocol

$$\dot{\xi}_i = -\sum_{j\in\mathcal{N}_i} \lambda_{ij}(\xi_i - \xi_j). \tag{6}$$

If we let $\xi = (\xi_1, \ldots, \xi_N)$ then we see that we can write $\dot{\xi} = \Lambda\xi$ where Λ is an interconnection matrix [2]. Thus if Λ is connected then $\xi_i \to \xi^*$ for some constant ξ^*. This key result has been the basis of much of the literature on the consensus problem.

Because we can associate a consensus protocol with an interconnection matrix, which in turn can be associated with a directed, weighted graph, we can likewise associate a directed, weighted graph with a consensus problem if we view the graph as having nodes that are integrators acting on the difference between the node's "value" ξ_i and those "values" ξ_j of its neighbors, weighted by the interconnection weight between the two nodes. Thus we are led to call such a graph a *consensus network*. In the next section we generalize this notion to the case where the weighting is defined by a real rational function.

3. CONSENSUS NETWORKS OVER THE REAL RATIONALS

We now consider graphs formed by nodes with multiple terminals that are connected by weighted arcs where the weights are represented as transfer functions taken from real rational functions analytic in the right half of the complex plane. These graphs are connected with the dynamic consensus protocol introduced earlier, and the dynamic Laplacian matrix is introduced. We then present our main result.

3.1 Terminals and Interconnections over the Real Rationals

Consider Fig. 3. The node has several terminals, each of which has a shared variable ξ_i, which is viewed as the node variable corresponding to the consensus variable of the overall graph. Each terminal also has its own associated variable q_{ij}, which can be considered an "outgoing" flow from node n_i to node n_j. We will typically associate one or more terminals with an "input" variable, denoted here as q_i^{in}. In the general case we view a node as implementing a transfer function that produces the node variable resulting from the incoming and outgoing flows. Thus, as depicted in Fig. 3 we have

$$\Xi_i(s) = H^i(s)\begin{bmatrix} Q_i^{in}(s) \\ Q_{i1}(s) \\ \vdots \\ Q_{ip}(s)\end{bmatrix}, \tag{7}$$

[2] Note that if $x_i \in \mathbb{R}^n$ then such a consensus protocol has the resulting form $\dot{x} = (\Lambda \otimes I_n)x$ where $\otimes$ is the Kronecker product.

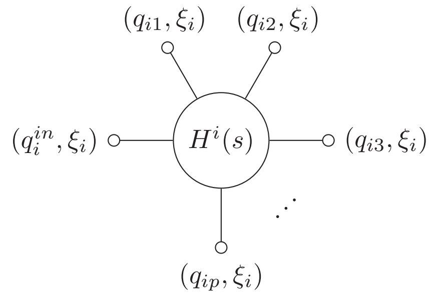

Fig. 3. Node element.

where we use upper-case notation to denote the Laplace transform of a lower-case variable. In the remainder of this paper we consider the special case where each node is an integrator. Thus we can write

$$\dot{\xi}_i(t) = q_i^{in}(t) - \sum_{j\in\mathcal{N}_i} q_{ij}(t) \tag{8}$$

or equivalently

$$s\Xi_i(s) = Q_i^{in}(s) - \sum_{j\in\mathcal{N}_i} Q_{ij}(s) \tag{9}$$

Note that the use of the negative sign in (8) and (9) is part of the definition of the node's transfer function and is motivated by the idea of "flows" into and out of the node. Additionally, the neighbors of a node n_i are again defined in terms of those other nodes to which the "flow" is directed.

Now consider the interconnection shown in Fig. 4. We define the flow out of n_i into node n_j as $q_{ij}(t)$ where the "amount" of flow is defined in the complex frequency domain by

$$Q_{ij}(s) = \begin{bmatrix}\lambda_{ij}^S(s) & -\lambda_{ij}^C(s)\end{bmatrix}\begin{bmatrix}\Xi_i(s) \\ \Xi_j(s)\end{bmatrix}, \tag{10}$$

$$= \lambda_{ij}^S(s)\Xi_i(s) - \lambda_{ij}^C(s)\Xi_j(s), \tag{11}$$

where $\lambda_{ij}(s) = \begin{bmatrix}\lambda_{ij}^S(s) & -\lambda_{ij}^C(s)\end{bmatrix}$ is the multi-dimensional real rational weight defined above in (2), with self-correction weight $\lambda_{ij}^S(s)$ and cross-correction weight $\lambda_{ij}^C(s)$. Arcs defined in this way can be thought of as implementing multi-input, single-output "transfer functions" between the nodes that they connect. As such, arcs induce the flows $Q_{ij}(s)$ as a weighted function of the associated node variables Ξ_i and Ξ_j.

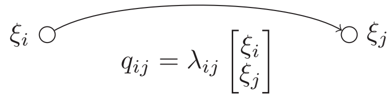

Fig. 4. Connection element.

3.2 Graphs over the Real Rationals

Now let us consider graphs comprised of the terminal and interconnection elements described in the previous subsection for the special case when the nodes are integrators. We will call such a graph a *dynamic graph* or a *dynamic consensus network*. Combining (9) and (10) we get

$$s\Xi_i(s) = Q_i^{in}(s) - \sum_{j\in\mathcal{N}_i}\left[\lambda_{ij}^S(s)\Xi_i(s) - \lambda_{ij}^C(s)\Xi_j(s)\right]. \tag{12}$$

Thus, if we define the vectors

$$\Xi(s) = [\Xi_1(s)\ \Xi_2(s)\ \ldots\ \Xi_N(s)]^T, \quad (13)$$

$$Q^{in}(s) = \left[Q_1^{in}(s)\ Q_2^{in}(s)\ \ldots\ Q_N^{in}(s)\right]^T, \quad (14)$$

then we can write

$$s\Xi(s) = Q^{in}(s) - L(s)\Xi(s), \quad (15)$$

the matrix $L(s) = [L_{ij}(s)]$ is given as

$$L_{ij}(s) = \begin{cases} \sum_{k\in\mathcal{N}_i} \lambda_{ik}^S(s) & i = j, \\ -\lambda_{ij}^C(s) & i \neq j \text{ and } (i,j) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

We call $L(s)$ defined in this way a *dynamic Laplacian*. For illustration, consider the graph topology shown in Fig. 2. For this system the dynamic Laplacian has the form shown in (16).

4. MAIN RESULTS

To state the main results, it will be useful to define the set $\mathbb{S} = \{s : \mathrm{Re}(s) \geq 0,\ s \neq 0\}$ and the following.

Definition 4.1. $L(s)$ is a *dynamic interconnection matrix* if it satisfies the following properties:

(1) $L(0)$ is an interconnection matrix.
(2) The elements of $L(s)$ have no poles in the closed right half of the complex plane.
(3) The diagonal elements of $L(s)$ satisfy the following positivity condition: for all $s \in \mathbb{S}$, $\mathrm{Re}\{L_{ii}(s)\} > 0$.
(4) $L(s) = [L_{ij}(s)]$ satisfies the following diagonal dominance condition: for all $s \in \mathbb{S}$,

$$\mathrm{Re}\{L_{ii}(s)\} > \sum_{j\neq i} |L_{ij}(s)|.$$

Remark 4.1. Although condition 3 implies that $L_{ii}(s)$ is passive, the diagonal dominance condition does not require L_{ij}, $j \neq i$ to be passive.

Definition 4.2. A dynamic interconnection matrix is *connected* if $L(0)$ is connected.

We may now state the following result:

Theorem 4.1. Consider the system

$$s\Xi(s) = L(s)\Xi(s), \quad (17)$$

where $L(s)$ is a connected dynamic interconnection matrix. Then

(1) $s = 0$ is a distinct solution of $\det[sI + L(s)] = 0$.
(2) All non-zero solutions of $\det[sI + L(s)] = 0$ are in the open left half of the complex plane.
(3) From any arbitrary set of initial conditions the node variables come to consensus, meaning $\xi_i(t) \to \xi^*$ for some constant ξ^*.

Proof 4.1. See the appendix.

5. ILLUSTRATION

We illustrate our results for the example of modeling thermal process in building. We first define the dynamic equations associated with such processes. Then we give a specific instantiation for a hypothetical four-room example.

5.1 Thermal Processes in a Building

It is common to model the thermal processes in a building using what is called a 3R2C model for the heat flow between rooms (see Xu and Wang (2007)). The basic model is depicted in Fig. 1 for the case when there are no heat inputs to the system.

From Fig. 1, the node equation can be written as

$$C_i^r \frac{dT_i}{dt} = -q_{ij}(t) \quad (18)$$

or

$$sC_i^r T_i(s) = -Q_{ij}(s). \quad (19)$$

Likewise, the heat flows in Fig. 1 can be derived to be:

$$\begin{bmatrix} Q_{ij}(s) \\ Q_{ji}(s) \end{bmatrix} = \frac{1}{B_{ij}(s)} \begin{bmatrix} A_{ij}(s) & -D_{ij}(s) \\ -D_{ij}(s) & A_{ji}(s) \end{bmatrix} \begin{bmatrix} T_i \\ T_j \end{bmatrix}, \quad (20)$$

where

$$\begin{aligned} A_{ij}(s) &= 1 + a_1^{ij}s + a_2^{ij}s^2, \\ A_{ji}(s) &= 1 + a_1^{ji}s + a_2^{ji}s^2, \\ B_{ij}(s) &= b_0^{ij} + b_1^{ij}s + b_2^{ij}s^2, \\ D_{ij}(s) &= 1 + d_1^{ij}s + d_2^{ij}s^2; \end{aligned}$$

and

$$\begin{aligned} a_1^{ij} &= C_4R_5 + C_2R_3 + C_2R_5, \\ a_2^{ij} &= C_4C_2R_5R_3, \\ a_1^{ji} &= C_4R_1 + C_4R_3 + C_2R_1, \\ a_2^{ji} &= C_4C_2R_3R_1, \\ b_0^{ij} &= R_5 + R_3 + R_1, \\ b_1^{ij} &= C_4R_5R_1 + C_2R_3R_1 + C_2R_5R_1 + C_4R_5R_3, \\ b_2^{ij} &= C_4C_2R_5R_3R_1 \\ d_1^{ij} &= d_2^{ij} = 0; \end{aligned}$$

Combining (20) and (19) gives

$$\begin{bmatrix} sC_i^r T_i(s) \\ sC_i^r T_j(s) \end{bmatrix} = -\frac{1}{B_{ij}(s)} \begin{bmatrix} A_{ij}(s) & -D_{ij}(s) \\ -D_{ij}(s) & A_{ji}(s) \end{bmatrix} \begin{bmatrix} T_i \\ T_j \end{bmatrix}. \quad (21)$$

Equation (21) defines the relationship between the temperatures in two rooms using the 3R2C model. Now suppose that we have several rooms in a building. Interpreting each room as an integrating room node in a graph, then we could write

$$sT_i(s) = \frac{1}{C_i^r}[Q_i^{in}(s) - \sum_{j\in\mathcal{N}_i} Q_{ij}(s)], \quad (22)$$

with the heat flows between any pair of nodes given by (20). Note that (20) implies that for any two adjacent nodes (rooms) we will have an arc from node n_i to node n_j and another arc from node n_j to node n_i. But, it would not be correct to consider the resulting graph to be undirected, because the resulting weights are different in each direction.

5.2 Hypothetical Four Room Example

Consider the four room example shown in Fig. 5. We assume there are no inputs or ambient temperature. Intuitively, we know that in this case the room temperatures will come to the same equilibrium, or in other words, the node variables reach consensus. We will confirm that this intuitive result can be predicted from the dynamic Laplacian that defines the building thermal dynamics. Figure 6 depicts the same four room example as a graph. In this figure the dynamic weights are given by

$$\lambda_{ij}(s) = \left[\left(\frac{A_{ij}}{B_{ij}}\right)_w + \frac{1}{R_{ij}^d} \qquad - \left(\frac{1}{B_{ij}}\right)_w - \frac{1}{R_{ij}^d} \right], (23)$$

where the terms $A_{ij}(s), A_{ji}(s), B_{ij}(s)$, and $D_{ij} = 1$ are defined by the 3R2C model given above, the subscript w

$$L(s) = \begin{bmatrix} \sum_{j=2,3} \lambda_{1j}^S(s) & -\lambda_{12}^C(s) & -\lambda_{13}^C(s) & 0 & 0 & 0 \\ -\lambda_{21}^C(s) & \sum_{j=1,4} \lambda_{2j}^S(s) & 0 & -\lambda_{24}^C(s) & 0 & 0 \\ 0 & 0 & \lambda_{35}^S(s) & 0 & -\lambda_{35}^C(s) & 0 \\ 0 & 0 & 0 & \lambda_{45}^S(s) & -\lambda_{45}^C(s) & 0 \\ 0 & -\lambda_{52}^C(s) & -\lambda_{53}^C(s) & 0 & \sum_{j=2,3,6} \lambda_{5j}^S(s) & -\lambda_{56}^C(s) \\ 0 & 0 & 0 & -\lambda_{64}^C(s) & 0 & \lambda_{64}^S(s) \end{bmatrix}. \tag{16}$$

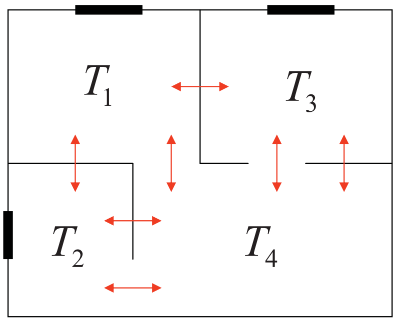

Fig. 5. Hypothetical four room example.

denotes thermal pathways through walls, and the terms $\frac{1}{R_{ij}^d}$ denote pathways through doors (note that not every interconnection has a pathway through a door). It is readily verified that the associated dynamic Laplacian, given in (24) is in fact a connected dynamic interconnection matrix. Thus we expect all room temperatures to converge to the

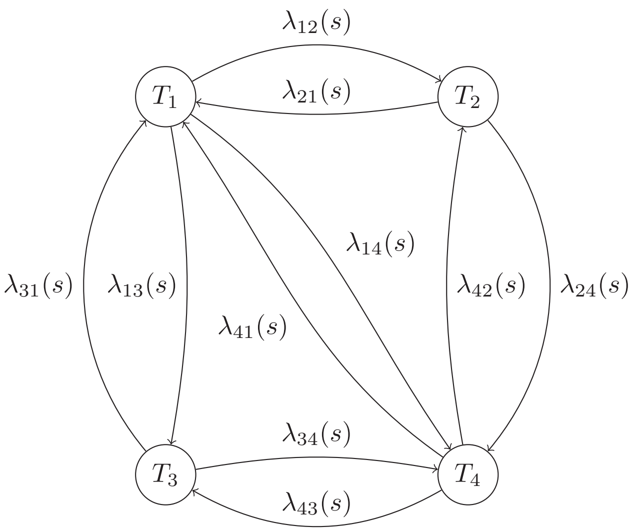

Fig. 6. Hypothetical four room example as a graph.

same constant. This is confirmed in Fig. 7, which shows a simple simulation of the dynamic consensus protocol (3) for a nominal set of parameters available from the authors upon request. Note that in this simulation, the initial condition for (4) was taken to be zero, which is equivalent to setting the temperature of the walls to zero. Thus the value of the final temperature is less than the initial room temperatures.

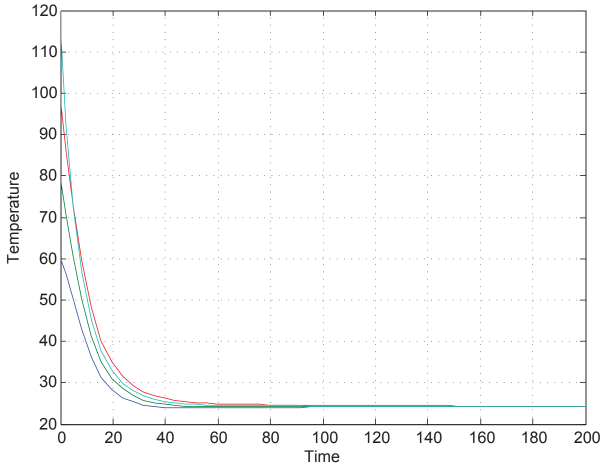

Fig. 7. Simulation of four room building. Each line corresponds to a room temperature (temperatures in degrees C).

6. CONCLUSIONS

In this paper we have generalized recent results on consensus networks with static weights by introducing the idea of consensus networks with real rational weights. We defined the idea of a dynamic Laplacian and a dynamic interconnection matrix and showed that when the dynamic Laplacian was the negative of a connected dynamic interconnection matrix then the dynamic consensus network achieves consensus. A representative example was given to illustrate the ideas. In future work we will extend these notions to consider controllability in the non-autonomous case as well as to consider observability and identifiability.

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APPENDIX

The proof of Theorem 4.1 depends on the Gershgorin circle theorem, and a characterization of solutions for systems defined using a behavioral approach.

Lemma A.1. (Gershgorin cricle theorem). (see. e.g. Golub and Van Loan (1996)) Let $A \in \mathbb{C}^{n\times n}$ with elements a_{ij}. Let $R_i = \sum_{j\neq i} |a_{ij}|$ and define $D(a_{ii}, R_i)$ to be the closed disk centered at a_{ii} with radius R_i. Then every eigenvalue of A lies within at least one of the disks $D(a_{ii}, R_i)$.

Lemma A.2. (Polderman and Willems (1998)). Given $m\times m$ polynomial matrix $P(s)$, let λ_i, $i = 1, \cdots, N$ be the distinct roots of $\det P(s)$ with multiplicity n_i. Then the solutions to

$$\begin{bmatrix} \frac{1}{R_{14}^d} + \sum_{j=2,3} \frac{A_{1j}}{B_{1j}} & -\frac{1}{B_{12}} & -\frac{1}{B_{13}} & -\frac{1}{R_{14}^d} \\ -\frac{1}{B_{21}} & \frac{1}{R_{24}^d} + \sum_{j=1,4} \frac{A_{2j}}{B_{2j}} & 0 & -\frac{1}{B_{24}} - \frac{1}{R_{24}^d} \\ -\frac{1}{B_{31}} & 0 & \frac{1}{R_{34}^d} + \sum_{j=1,4} \frac{A_{3j}}{B_{3j}} & -\frac{1}{B_{34}} - \frac{1}{R_{34}^d} \\ -\frac{1}{R_{41}^d} & \frac{-1}{B_{42}} - \frac{1}{R_{42}^d} & -\frac{1}{B_{43}} - \frac{1}{R_{43}^d} & \frac{1}{R_{41}^d} + \frac{1}{R_{42}^d} + \frac{1}{R_{43}^d} + \sum_{j=2,3} \frac{A_{4j}}{B_{4j}} \end{bmatrix} \tag{24}$$

$$P\left(\frac{d}{dt}\right)x(t) = 0$$

are of the form

$$x(t) = \sum_{i=1}^{N} \sum_{j=0}^{n_i-1} \alpha_{ij} t^j e^{\lambda_i t}$$

where α_{ij} are constants.

Proof of Theorem 4.1:

1) This follows directly from the fact that $L(0)$ is connected.

2) Suppose, to the contrary, there exists s^* such that $\det[s^* I + L(s^*)] = 0$ and $s^* \in \mathbb{S}$. Since $L(s)$ is a dynamic interconnection matrix, $L(s^*)$ is finite. Note that s^* is an eigenvalue of $-L(s^*)$. Define $D(a, R)$ to be the closed disk in the complex plane centered at a, with radius R. By the Gershgorin circle theorem, the eigenvalues of $-L(s^*)$ are contained in the disks $D(-L_{ii}(s^*), R_i)$ where $R_i = \sum_{j\neq i} |L_{ij}(s^*)|$. By the positivity and diagonal dominance conditions of a dynamic interconnection matrix, none of these disks intersect $\mathbb{S}$, which leads to a contradiction. Thus there exists no solution in $\mathbb{S}$.

3) Given 1), this claim is proven if we can show that no solution to (17) exists with modes in $\mathbb{S}$. Let $B_i(s)$ be the least common multiple of the denominators of the ith row of $L(s)$, and let $B(s) = \mathrm{diag}\,[B_1(s)\ B_2(s)\ \cdots\ B_m(s)]$. Let $C(s) = B(s)L(s)$. Note that both $B(s)$ and $C(s)$ are polynomial matricies. Then it can be shown (see e.g. Polderman and Willems (1998)) that the solutions to (17) are given by $\xi(t)$ that satisfy

$$(sB(s) + C(s))|_{s=\frac{d}{dt}}\, \xi(t) = 0$$

and by Lemma A.2 the allowable time domain modes of $\xi(t)$ are given by the roots of $\det(sB(s) + C(s))$. Note that since $B(s)$ has no roots in the closed right half plane, the solutions to $\det(sB(s) + C(s)) = 0$ on $\mathbb{S}$ are identical to the solutions of $\det\left(sI + B^{-1}(s)C(s)\right) = \det(sI + L(s)) = 0$. However, by part 2, there are no solutions in $s \in \mathbb{S}$ which proves the claim.

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